

# ADVANCES IN QUANTUM AND BRAIDED GEOMETRY

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We demonstrate our recent general results on the Casimir construction and moduli space of all bicovariant calculi by means of some detailed examples, including finite-difference and 2-jet calculi on  $\mathbb{R}^n$  and full details of the Casimir construction of the 4D calculus of  $SU_q(2)$ . We likewise demonstrate our previous general constructions with T. Brzezinski of quantum group gauge theory with examples of such nonuniversal differential calculi on spacetime. We outline a notion of quantum homotopy of a quantum space. We indicate a possible application to classical integrable systems.

## 1 Introduction

There has been a lot of progress in recent years towards developing some form of q-geometry appropriate to q-deformed spaces. There are two main strands; the approach based on braided categories and braid statistics (braided geometry) due to the author, see [1][2][3], and the more conventional approach within noncommutative geometry based on abstract differential calculus for quantum groups[4]. True differential geometry in the latter setting, meaning connections, gauge theory, q-monopoles, etc. (with a quantum group fiber and quantum space base) is due to T. Brzezinski and the author[5]. By now it has subsequently been further developed by several authors. We emphasise the noncommutative version but also mention progress in the braided version and beyond it.

Until now, emphasis has been on attempts to find q-deformations of classical calculi etc., or their variants[6], to the extent that this is possible for each quantum group. The full classification problem or construction of the ‘moduli space’ of *all* possible bicovariant differential calculi on general classes of quantum groups has remained largely unexplored. Some progress in this ‘moduli space’ direction was made recently in [7], which we recall briefly in Section 2. The main concept is the notion of a *coirreducible calculus*, without which one cannot begin a classification. We also recall from [7] a new ‘direct’ construction of the quantum tangent spaces of bicovariant calculi from Casimir elements of the quantum enveloping algebra. We then provide some detailed

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examples of the classification for  $\mathbb{R}^n$ , and the Casimir construction on  $SU_q(2)$ , which are the new results beyond [7] of this section. The Casimir construction works uniformly in the quantum and classical cases; we see now why the dimension jumps when  $q \neq 1$ .

In Section 3 we give some detailed computations of quantum group gauge theory using these general nonuniversal calculi applied to the general framework of [5]. It is not hard to come up with some of these examples by hand, but we fit them now into the uniform general quantum group gauge theory. We also indicate an application of quantum group gauge theory even with trivial bundles, to define the quantum fundamental group of an algebra.

Although there are some modest new results, we aim here at a fresh self-contained treatment of the geometrical theory, as a readable introduction to [5][3][2][7] [8]. Some open problems will be emphasised as well. For basic results and notations for quantum groups themselves, see [1].

## 2 Moduli of Quantum Tangent Spaces

In this section we look at the first layer of geometry, the choice of *differentiable structure*. We are interested in our noncommutative algebra  $A$  being like the coordinate ring on a general manifold, i.e. not just a quantum group, though this is the case best understood at the present time. As usual, we do this through the notion of ‘tangent space’ or (equivalently) ‘1-forms’  $\Omega^1$ . By specifying one of these, we specify in effect the differentiable structure (which is the choice of diffeomorphism class of a manifold for a given topological class). The algebra by itself should be thought of as topological object, needing more structure to specify differentiation. What is interesting is that whereas the additional structure is fairly unique classically (at least when we have left and right invariance under a group structure), one really should not expect this in the noncommutative case, as we explain in Section 2.1. I.e. we must think much more in terms of the moduli space of calculi than we do classically. It is a new degree of freedom in quantum geometry, which we can develop field equations for, integrate over in q-quantum gravity, etc.

### 2.1 Nonuniqueness of bicovariant calculi

The fundamental reason for the above-mentioned non-uniqueness is as follows. In terms of 1-forms, we need to specify a vector space  $\Omega^1$  on which  $A$  acts from the left and the right (classically, one may multiply a 1-form by a function from the left or right). In addition, we need a linear map  $d : A \rightarrow \Omega^1$  which turns ‘functions’ into ‘1-forms’. The natural axioms are:

1.  $\Omega^1$  an  $A$ -bimodule
2. The Leibniz rule  $d(ab) = (da).b + a.(db)$  for all  $a, b \in A$
3. Surjectivity in the form  $\Omega^1 = \text{span}\{adb\}$

This is called the choice of *first order differential calculus*. It is more or less the minimal structure needed for some kind of differential geometry. Note that when  $A$  is not commutative, it makes no sense to assume that

$$a.(db) = (db).a \quad (1)$$

as we would assume classically. This is because such an assumption and the Leibniz rule would imply  $d(ab) = d(ba)$ , which is not reasonable when our algebras are non-commutative. This is the reason that we require  $\Omega^1$  to be an  $A$ -bimodule. On the other hand, the assumption (1) of a ‘commutative calculus’ is highly restrictive classically, and when we relax it we have a much larger range of possibilities than in the classical case.

When  $A$  is a Hopf algebra it is natural to focus on differential calculi which are covariant under left or right translations on the ‘group’. Classically, this invariance condition fixes the differentiable structure uniquely as that obtained by translation from the differentiable structure at the identity. One usually identifies the tangent space at the identity with the Lie algebra  $\mathfrak{g}$  and extends it to  $G$  via left-invariant vector fields  $\tilde{\xi}$  for  $\xi \in \mathfrak{g}$ . These then transform under right translation via the adjoint action. Likewise, at the level of ‘1-forms’. One may impose similar conditions in the quantum case, that  $\Omega^1$  is also an  $A^*$ -bimodule (or more precisely an  $A$ -bicomodule) in a manner compatible with the  $A$ -bimodule and  $d$  structures. This is called a *bicovariant differential calculus*. Being the most restrictive, we might hope to have uniqueness at least in this case.

We recall first that every quantum group has a counit  $\epsilon : A \rightarrow \mathbb{C}$  which, classically, denotes evaluation at the group identity. The space  $\ker \epsilon \subset A$  has a natural action of  $A$  by multiplication from the left and of  $A^{\text{op}}$  by the quantum coadjoint action  $\text{Ad}^*$ . The two actions form an action of the quantum double  $D(A)$ . Then, after a little analysis, one finds cf[4]:

**Proposition 2.1** *A bicovariant  $\Omega^1$  must be of the form  $\Omega^1 = V \otimes A$*

$$a.(v \otimes b) = a_{(1)} \triangleright v \otimes a_{(2)} b, \quad (v \otimes b).a = v \otimes ba, \quad da = (\pi \otimes \text{id})(1 \otimes a - a_{(1)} \otimes a_{(2)})$$

for some vector space  $V$  which is a quotient of  $\ker \epsilon$  as a  $D(A)$ -module, i.e.  $V = \ker \epsilon / M$  where  $M \subset \ker \epsilon$  is some  $D(A)$ -submodule. Here  $\pi : \ker \epsilon \rightarrow V$  denotes the canonical projection and  $\triangleright$  denotes left multiplication by  $A$  projected down to  $V$ .

The vector space  $V$  is called the space of left-invariant 1-forms, and its dimension is called the dimension of the differential calculus. For  $A = SU_q(2)$  the lowest dimension bicovariant calculus when  $q \neq 1$  has dimension 4 (not 3!) and was found in [4]. Woronowicz found in fact two calculi of dimension 4 given by a similar construction. More generally, for the ABCD series of semisimple Lie groups there is a natural R-matrix construction[9] for a bicovariant calculus of dimension  $n^2$  (where  $G \subset M_n$ ), and some variants of it which are classified case-by-case in [6].

It may seem therefore that (up to some variants related to a choice of roots) there is one natural bicovariant calculus for each of the standard quantum groups  $G_q$ . This is misleading, however. It was shown in [10] that there is at least a whole ‘algebra’ of calculi associated to the non-trivial elements  $\alpha$  of the algebra  $Z^*(A)$  of  $\text{Ad}^*$ -invariant elements. Classically, these are the class functions on the group  $G$  (the functions which are constant on each conjugacy class). Explicitly,

$$\alpha \in Z^*(A) = \{a \in A | a_{(1)}(Sa_{(3)}) \otimes a_{(2)} = 1 \otimes a\}, \quad V = \ker \epsilon / \ker \epsilon(\alpha - (\epsilon(\alpha) + 1)) \quad (2)$$

defines the calculus, where  $(\text{id} \otimes \Delta) \circ \Delta a = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$  is the notation. Moreover,  $d$  then has the inner form [10]

$$da = a \cdot \omega_\alpha - \omega_\alpha \cdot a, \quad \omega_\alpha = (d\alpha_{(1)})S\alpha_{(2)} \in \Omega^1, \quad (3)$$

where  $S$  is the antipode or ‘linearised inversion’ operator on  $A$ . This shows that the entire construction is not possible classically, since (1) would force  $da = 0$ . For the standard quantum groups  $G_q$ , it is also shown in [10] that  $Z^*(G_q)$  is commutative with rank- $g$  algebraically independent generators  $\{\alpha_i\}$  (this is proved by studying the braided group version of  $G_q$ ). Only one of these, the  $q$ -trace, is relevant to the natural  $n^2$ -dimensional calculus of [9], which appears after making further quotients of  $V$  [10].

For example, for the quantum group  $SU_q(2)$  with the standard matrix of generators  $\mathbf{t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , we take

$$\alpha = \frac{qa + q^{-1}d}{(q - q^{-2})(q - 1)}. \quad (4)$$

Here  $d$  in (3) has a classical remnant as  $q \rightarrow 1$  because the normalisation of  $\alpha$  goes to  $\infty$ , yielding a certain bicovariant differential calculus on  $SU_2$ .

## 2.2 Construction of invariant quantum tangent spaces

Even the family of calculi associated to the algebra  $Z^*(A)$  does not exhaust the bicovariant calculi on a typical quantum group. To get some feel for the entire moduli space, i.e. to classify all calculi, it is convenient to concentrate on the *invariant quantum tangent space* associated to every bicovariant calculus. This is the ‘tangent space at the identity’  $L = V^*$  and can be characterised as follows. First of all, note that  $A^*$  (defined in a suitable way) will generally also be a Hopf algebra, with its own counit  $\epsilon$ . This is the self-duality of the axioms of a Hopf algebra. Moreover,  $\ker \epsilon \subset A^*$  is acted upon by the quantum double  $D(A^*)$  by

$$x \triangleright y = \text{Ad}_x(y) = x_{(1)}ySx_{(2)}, \quad a \triangleright y = \langle a, y_{(1)} \rangle y_{(2)} - 1 \langle a, y \rangle, \quad \forall y \in \ker \epsilon \subset A^* \quad (5)$$

for the action of  $a \in A$  and  $x \in A^*$ . Here  $\text{Ad}$  is the quantum adjoint action, while the action of  $A$  is rather strange. Classically,  $A^* = U(g)$  and  $\ker \epsilon$  denotes any product of Lie algebra

elements. The quantum adjoint action becomes the usual coadjoint action restricted to  $\ker \epsilon$ , and the action of  $A = \mathbb{C}(G)$  becomes to lowest degree

$$a \triangleright \xi = a(e)\xi, \quad a \triangleright (\xi\eta) = a(e)\xi\eta + \langle a, \xi \rangle \eta + \langle a, \eta \rangle \xi$$

etc for  $\xi, \eta \in g$ . The action on higher degree elements of  $U(g)$  is more complicated, as determined by the coproduct  $\Delta$ .

**Proposition 2.2** *The possible invariant quantum tangent spaces are precisely the subspaces  $L \subset \ker \epsilon \subset A^*$  which are subrepresentations of the quantum double  $D(A^*)$  acting on  $\ker \epsilon$ . Explicitly, this means  $L$  such that:*

$$L \subset \ker \epsilon \subset A^*, \quad \text{Ad}_x(L) \subset L, \quad (\Delta - \text{id} \otimes 1)(L) \subset A^* \otimes L \quad (6)$$

for all  $x \in A^*$ .

The explicit version (6) works (more or less equivalently) with everything in terms of the Hopf algebra structure of  $A^*$  rather than mentioning the quantum double. Also note that although these quantum tangent spaces  $L$  are sometimes called ‘quantum Lie algebras’[4], they do not usually have enough structure to qualify for a Lie algebra of any kind. In nice cases where they do have enough structure, they are more properly formulated as *braided-Lie algebras*[11]. Moreover, for any Hopf algebra  $A$ , the associated differentiation operators  $\partial_x : A \rightarrow A$  defined by  $\partial_x(a) = (\langle x, \rangle \otimes \text{id}) \circ \text{d}$  obey a *braided Leibniz rule*[7]

$$\partial_x(ab) = (\partial_x a)b + a_i \partial_{x^i} b \quad (7)$$

for all  $a, b \in A$  and  $x \in L$ . Here  $\Psi^{-1}(a \otimes x) \equiv x^i \otimes a_i$  is our notation (summation understood) and  $\Psi$  is the braiding of  $L$  with  $A$  as spaces on which the quantum double  $D(A^*)$  acts. The elements of  $L$  define in this way the ‘braided left-invariant vector fields’ associated to a bicovariant calculus.

This quantum tangent space point of view is developed in detail in [7], where some general classification theorems are then obtained. Firstly, one should introduce the notion of *coirreducible calculus*, which we define as corresponding to an irreducible quantum tangent space. It turns out that the classical case  $A = \mathbb{C}(G)$  is rather singular and is actually the hardest; we say more about it in Section 2.3. The finite group classical case is easier and it is known already[12] that natural bicovariant calculi are obtained from nontrivial conjugacy classes in  $G$ . It is proven in [7] that these are in fact coirreducible, and that coirreducible calculi on  $A = \mathbb{C}(G)$  are in 1-1 correspondence with nontrivial conjugacy classes. The span of the conjugacy class is  $L$ . Also, taking the view of noncommutative geometry that the group algebra  $A = \mathbb{C}G$  for a finite group can be viewed as ‘like’  $\mathbb{C}(\hat{G})$  (here  $\hat{G}$  need not exist, however), one finds[7] that the coirreducible

calculi in this case are classified by pairs consisting of an irreducible representations  $\rho$  and a continuous parameter in the projective space  $\mathbb{C}P^{\dim \rho - 1}$ . They have dimension  $(\dim \rho)^2$ . Finally, for a strict semisimple quantum group (with a universal R-matrix  $\mathcal{R}$  obeying some further strict conditions) one can again classify the coirreducible calculi[7]; they turn out again to correspond to the irreducible representations  $\rho$  of  $g$  and have dimension  $(\dim \rho)^2$ . This applies essentially to the standard deformations  $G_q$  for generic  $q$ , up to some variants. In the case of  $SU_q(2)$ , one knows that the quantum double of  $U_q(su_2)$  can be identified when  $q \neq 1$  with some version of the  $q$ -Lorentz group. Hence its possible invariant quantum tangent spaces are the  $q$ -deformation (and projection to  $\ker \epsilon$ ) of the subrepresentations of the particular representation consisting of the Lorentz group acting on the space of functions on  $SU_2$  by left and right group translation. The lowest possible coirreducible calculus is therefore 4-dimensional (the action on Minkowski space). This is the tensor square of the spin 1/2 representation of  $SU_2$ . The full analysis[7] shows that generically (up to some uniform choices), the tensor square of each spin  $j > 0$  irreducible representation of  $SU_2$  occurs, just once. So there is a natural 9-dimensional spin 1 calculus, a 16 dimensional spin 3/2 calculus, etc.

Hence the classical  $q = 1$  theory is more like the finite group case, while the  $q \neq 1$  theory is more like the finite group-dual case. In all cases, there is an entire moduli space of calculi. One may endow the moduli space with a topology, and introduce natural operations on it. One of them, in [7], associates to every bicovariant calculus a ‘dual’ or ‘mirror’ calculus of a quite different form but on the same quantum group.

Also introduced in [7] is a natural construction for bicovariant quantum tangent spaces which is ‘dual’ to the  $\alpha$  construction in Section 2.1. It associates a quantum tangent space to any non-trivial element in the centre  $Z(A^*)$ . This is the algebra of elements of  $A^*$  fixed under the quantum adjoint action.

**Proposition 2.3** [7] *For any  $c \in Z(A^*)$ ,*

$$L = \text{span}\{x_a = \langle a, c_{(1)} \rangle c_{(2)} - \langle a, c \rangle | a \in \ker \epsilon \subset A\} \quad (8)$$

*defines an invariant quantum tangent space.*

We now compute the case  $A = SU_q(2)$  and  $A^* = U_q(su_2)$  in detail. We take the latter in its standard form with generators  $q^{\frac{H}{2}}, X_{\pm}$ , relations, coproduct and counit

$$\begin{aligned} q^{\frac{H}{2}} X_{\pm} q^{-\frac{H}{2}} &= q^{\pm 1} X_{\pm}, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad \Delta q^{\frac{H}{2}} = q^{\frac{H}{2}} \otimes q^{\frac{H}{2}}, \quad \epsilon q^{\frac{H}{2}} = 1 \\ \Delta X_{\pm} &= X_{\pm} \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes X_{\pm}, \quad \epsilon X_{\pm} = 0. \end{aligned} \quad (9)$$

There is a well-known quadratic  $q$ -Casimir which we take in a certain normalisation and offset by a multiple of 1:

$$c_q = \frac{C - (q + q^{-1})}{(q - q^{-2})(q - 1)}; \quad C = q^{H-1} + q^{-H+1} + (q - q^{-1})^2 X_+ X_- . \quad (10)$$

**Proposition 2.4** *The invariant quantum tangent space of  $SU_q(2)$  generated by the  $q$ -Casimir  $c_q$  in (10) via Proposition 2.3 is 4-dimensional and coincides with the braided Lie algebra  $\underline{gl}_{2,q}$  in [11] when  $q \neq 1$ . Explicitly,  $L = \text{span}\{x_{a-1}, x_b, x_c, x_{d-1}\}$ , where*

$$\begin{aligned} x_{a-1} &= \frac{q+1}{q-q^{-2}}(q^H - 1) + (q^{-1} - 1)c_q, & x_b &= \frac{q^{\frac{1}{2}}(q+1)(1-q^{-2})}{q-q^{-2}}q^{\frac{H}{2}}X_- \\ x_c &= \frac{q^{\frac{1}{2}}(q+1)(1-q^{-2})}{q-q^{-2}}X_+q^{\frac{H}{2}}, & x_{d-1} &= \frac{q^{-1}+1}{q^{-1}-q^2}(q^H - 1) + (q - 1)c_q. \end{aligned}$$

When  $q = 1$ , the space  $L$  is three-dimensional and coincides with the Lie algebra  $su_2$ .

**Proof** The duality pairing of  $U_q(su_2)$  with  $SU_q(2)$  is the fundamental representation

$$\langle \mathbf{t}, q^{\frac{H}{2}} \rangle = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \quad \langle \mathbf{t}, X_+ \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \langle \mathbf{t}, X_- \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Using this and the identity

$$\begin{aligned} \Delta C &= C \otimes q^H + q^{-H} \otimes C - (q + q^{-1})q^{-H} \otimes q^H \\ &\quad + (q - q^{-1})^2(X_+q^{-\frac{H}{2}} \otimes q^{\frac{H}{2}}X_- + q^{-\frac{H}{2}}X_- \otimes X_+q^{\frac{H}{2}}) \end{aligned}$$

which follows from (9), we compute the elements of  $L$  corresponding to  $a-1, b, c, d-1 \in \ker \epsilon \subset SU_q(2)$ . They are clearly linearly independent when  $q \neq 1$ . The products of these generators with general elements of  $SU_q(2)$  taken in the role of  $a \in \ker \epsilon$  in Proposition 2.3 do not give new elements of  $L$  because this 4-dimensional  $L$  is already known to be stable under the action of all  $a \in SU_q(2)$  in (5). Hence these elements are a basis for  $L$ . This vector space coincides with the braided Lie algebra  $\underline{gl}_{2,q}$  because this is identified[13] with a natural subspace of  $U_q(su_2)$ , according to

$$\begin{aligned} h &= \frac{q^{-1}}{q^2-1}(C - (q + q^{-1})q^H), & x &= q^{-\frac{3}{2}}q^{\frac{H}{2}}X_- \\ y &= q^{-\frac{3}{2}}X_+q^{\frac{H}{2}}, & \gamma &= \frac{q^{-1}}{q^2-1}(C - (q + q^{-1})). \end{aligned} \tag{11}$$

We recall that the braided Lie bracket in [13][11] is given by the quantum adjoint action restricted to this subspace. Explicitly,

$$\begin{aligned} [h, x] &= (q^{-2} + 1)x = -q^2[x, h], & [h, y] &= -(q^{-2} + 1)q^{-2}y = -q^{-2}[y, h] \\ [x, y] &= q^{-2}h = -[y, x], & [h, h] &= (1 - q^{-4})h, & [\gamma, \begin{Bmatrix} h \\ x \\ y \end{Bmatrix}] &= (1 - q^{-4}) \begin{Bmatrix} h \\ x \\ y \end{Bmatrix} \end{aligned} \tag{12}$$

The isomorphism (11) is valid for generic  $q \neq 1$ . By contrast, as  $q \rightarrow 1$  the elements  $x_{a-1}$  and  $x_{d-1}$  coincide and  $L$  becomes 3-dimensional in this special limit.  $\square$

The braided-Lie algebra  $\underline{gl}_{2,q}$  is known to be the quantum tangent space for the usual 4-dimensional bicovariant differential calculus on  $SU_q(2)$ . The Casimir construction in Proposition 2.3 from [7] provides a new and more direct, route. Moreover, in the moduli of all quantum

tangent spaces, only very few calculi (probably just one in the standard cases) will be close enough to the classical calculus to have the additional properties needed properly to be a ‘quantum’ or braided Lie algebra. Using the  $q$ -deformed quadratic Casimir of  $U_q(g)$  exactly picks out such a natural one, as the above example demonstrates in detail. Moreover, we achieve this without  $R$ -matrices; it works for general simple  $g$ , including the exceptional series.

### 2.3 Quantum tangent spaces on classical groups

In this section we specialise to the case  $A = \mathbb{C}(G)$ , where  $G$  is a classical Lie group. More precisely, we take  $A^* = U(g)$  where  $g$  is the Lie algebra of  $G$ . The classification of bicovariant calculi or their quantum tangent spaces in Proposition 2.2 is not known in any generality:

- It is an *unsolved problem in classical Lie theory to find all irreducible subspaces  $L \subset U(g)$  stable under  $\text{Ad}$  and  $\Delta - \text{id} \otimes 1$ .*

Of course, many possible  $L$  (not necessarily irreducible) are known. For example, we may take

$$L = g, \quad L = g + gg, \quad L = g + gg + ggg \quad (13)$$

etc. as subspaces of  $U(g)$ . Only the first of these obeys (1) but the rest are just as valid as soon as we relax this assumption. The construction in Proposition 2.3 works perfectly well in this classical case i.e. the  $q$ -deformed case in Section 2.2 has a smooth limit as  $q \rightarrow 1$ . When  $g$  is semisimple, there are rank- $g$  algebraically independent Casimirs, each with a natural bicovariant calculus.

We will focus in fact on the very simplest case, which nevertheless demonstrates the key points: we take  $G = \mathbb{R}$  the additive real line. Let us note that the possibility and applications of non-standard differential calculi on classical spaces such  $\mathbb{R}^n$  have already been emphasised in the papers of Müller-Hoissen and collaborators[14][15], although perhaps not as part of a systematic theory of bicovariant calculi and quantum tangent spaces as we present now.

Thus, we take  $A = \mathbb{C}[x]$  and  $U(\mathbb{R}) = \mathbb{C}[p]$ , i.e. functions in 1-variable with their linear coproducts. According to Proposition 2.2, we need to classify all subspaces  $L \subset \mathbb{C}[p]$  which obey  $\forall f(p) \in L$ ,

$$f(0) = 0, \quad (\Delta - \text{id} \otimes 1)f = f' \otimes p + f'' \otimes \frac{p^2}{2!} + f''' \otimes \frac{p^3}{3!} + \cdots \in \mathbb{C}[p] \otimes L. \quad (14)$$

An obvious family of choices is  $L = \text{span}\{p, p^2, \dots, p^n\}$  for any natural number  $n$ . For then all  $f \in L$  have the property that  $f^{(m)} = 0$  if  $m > n$ , ensuring the required condition. One can also write the second condition in (14) as  $f(p + \lambda) - f(\lambda) \in L$  for all  $\lambda$  (by evaluating against  $\lambda$ ). So

- An invariant quantum tangent space for  $\mathbb{R}$  means a subspace of functions vanishing at zero and closed under all projected translations (i.e. translations followed by subtraction of the value at zero).



We can also approach the construction of natural  $L$  through Proposition 2.3. All functions  $c(p)$  are central, so every function determines a calculus! This is therefore a new degree of freedom determined by an arbitrary function on  $\mathbb{R}$ . For a basis of  $\ker \epsilon \subset A$ , we take  $\{x, x^2, \dots\}$ . Then,

**Proposition 2.5** *Any function  $c(p)$  defines an invariant quantum tangent space on  $\mathbb{R}$ ,*

$$L = \text{span}\{p_n = c^{(n)}(p) - c^{(n)}(0), \quad n \in \mathbb{N}\}.$$

*The associated braided derivations  $\partial_{p_n}$  and their braiding with functions  $f(x)$  are*

$$\partial_{p_n} f = (c^{(n)}(\frac{d}{dx}) - c^{(n)}(0))f, \quad \Psi^{-1}(f \otimes p_n) = \sum_{m=0}^{\infty} p_{n+m} \otimes \frac{f^{(m)}}{m!}$$

The associated space of invariant 1-forms  $V$  is defined as  $L^*$  and is computed as follows. We start with  $\ker \epsilon \subset \mathbb{C}[x]$ , i.e. with  $f(x)$  such that  $f(0) = 0$ , and quotient out by the subspace of  $f$  such that

$$(\partial_{p_n} f)(0) = 0, \quad \forall n \in \mathbb{N}. \quad (15)$$

This follows from the duality  $\langle x^n, p^m \rangle = \delta_{n,m} m!$  between  $\mathbb{C}[x]$  and  $\mathbb{C}[p]$ . We then choose a basis  $\{e_i\}$  of  $L$  with dual basis  $\{f^i\}$  and define

$$d : A \rightarrow \Omega^1 = V \otimes A, \quad df = f^i \otimes \partial_{e_i} f. \quad (16)$$

**Proposition 2.6** *The function  $c(p) = p^{n+1}/(n+1)!$  in Proposition 2.5 yields the  $n$ -dimensional calculus with  $L = \text{span}\{p, p^2, \dots, p^n\}$ . The derivations, their braiding and exterior derivative are  $\partial_{p^m} f = f^{(m)}$  and*

$$\Psi^{-1}(f \otimes p^m) = \sum_{r=0}^{m-1} p^{m-r} \otimes f^{(r)} \frac{m!}{r!(m-r)!}, \quad df = \sum_{m=1}^n \frac{\pi(x^m)}{m!} \otimes f^{(m)}(x)$$

where  $\pi$  is the projection defined by  $x^{n+1} = 0$ .

**Proof** The  $p_n$  generators of  $L$  from Proposition 2.5 are  $p_{n-m+1} = p^m/m!$  for  $1 \leq m \leq n$  and  $p_m = 0$  for  $m > n$ . Hence we can take  $\{p, \dots, p^n\}$  as basis of  $L$ . Their action from the duality pairing between  $\mathbb{C}[x], \mathbb{C}[p]$  is by  $p$  as usual differentiation, while the braiding  $\Psi^{-1}$  follows immediately from the formula in Proposition 2.5. The space  $V$  consists of functions vanishing at zero modulo powers of  $x$  higher than  $x^n$ . Hence  $\{\pi(x), \pi(x^2)/2!, \dots, \pi(x^n)/n!\}$  is a dual basis for  $V$ . We then obtain  $df$  from (16).  $\square$

The case  $n = 1$  recovers the usual differential calculus on  $\mathbb{R}$ , while  $n > 1$  gives higher order *jet calculi* in which higher derivatives are regarded as first-order vector fields, but obeying a

braided Leibniz rule (7) with nontrivial  $\Psi^{-1}$ . The commutativity (1) also fails for  $n > 1$ . For example,  $c(p) = p^3/3!$  gives the 2-jet calculus with 1-forms obeying

$$df = (dx)f' + \omega f''; \quad \omega \equiv \frac{1}{2}(xdx - (dx)x), \quad fdx - (dx)f = 2\omega f', \quad f\omega = \omega f, \quad (17)$$

where we identify  $\pi(x) \otimes 1 = dx$  and  $\frac{1}{2}\pi(x^2) \otimes 1 = \omega$  as invariant forms. For another example, we formally make completions of our algebras to allow certain powerseries. Then

**Proposition 2.7** *For all  $\lambda \neq 0$ , the function  $c(p) = \lambda^{-2}e^{\lambda p}$  in Proposition 2.5 yields a 1-dimensional bicovariant differential calculus on  $\mathbb{R}$  with  $L = \{p_1 = \lambda^{-1}(e^{\lambda p} - 1)\}$ . The differentiation and braiding are*

$$\partial_{p_1} f = \frac{f(x+\lambda) - f(x)}{\lambda}, \quad \Psi^{-1}(f \otimes p_1) = p_1 \otimes f(x+\lambda).$$

**Proof** From Proposition 2.5, we find  $p_{n+1} = \lambda^n p_1$  so  $L$  is 1-dimensional. The braiding then follows at once from  $\Psi^{-1}$  in Proposition 2.5 and Taylor's theorem.  $\square$

Thus, the finite-difference operations introduced by Newton's tutor Barrow *are exact* differentiation with but respect to a non-standard differential calculus on  $\mathbb{R}$ . In fact, all coirreducible calculi on  $\mathbb{R}$  are of this form, parametrised by  $\lambda$ . And we only needed to relax (1) in order to allow them, without taking the limit  $\lambda \rightarrow 0$ . To see that (1) fails, we compute the 1-forms as follows. The vector space  $V$  consists of functions vanishing at 0 quotiented out by all  $f$  such that  $f(\lambda) = 0$ . This is essentially the functions divisible by  $x$  modulo those divisible by  $x(x-\lambda)$ , which is a 1-dimensional space. As a basis of  $V$  we can take  $\{\pi(x)\}$  since  $\langle p_1, x \rangle = 1$ . Then

$$df = \pi(x) \otimes \frac{f(x+\lambda) - f(x)}{\lambda} = (dx) \frac{f(x+\lambda) - f(x)}{\lambda}, \quad (18)$$

where we identify  $dx = \pi(x) \otimes 1$  to obtain the second expression. Similarly, we find that  $xdx - (dx)x = \pi(x^2) \otimes 1 = \lambda\pi(x) \otimes 1 = \lambda dx$ , so we see that (1) fails for this calculus. More generally, we deduce from this that

$$f \cdot dx - (dx) \cdot f = \lambda df \quad (19)$$

so that the calculus is of the inner form (3) discussed in [10].

The story is similar for  $\mathbb{R}^n$ . For example, on  $\mathbb{R}^2$  the function  $c(\mathbf{p}) = \lambda^{-2}e^{\lambda p} + \mu^{-2}e^{\mu q}$  (where  $\mathbf{p} = (p, q)$  and  $\lambda, \mu \neq 0$ ) gives the 2-dimensional calculus  $L = \text{span}\{p_{1,0}, p_{0,1}\}$  with

$$\begin{aligned} \partial_{p_{1,0}} f &= \frac{f(x+\lambda, y) - f(x, y)}{\lambda}, \quad \partial_{p_{0,1}} f = \frac{f(x, y+\mu) - f(x, y)}{\mu}, \quad df = (dx)\partial_{p_{1,0}} f + (dy)\partial_{p_{0,1}} f \\ xdy &= (dy)x, \quad ydx = (dx)y, \quad xdx - (dx)x = \lambda dx, \quad ydy - (dy)y = \mu dy. \end{aligned} \quad (20)$$

Of course, there are many other interesting functions  $c(p)$  one could take. Another interesting one is  $c(p) = \lambda^{-1}e^{\lambda p^2}$  yielding a 'Hermite differential calculus'. We truly have a new field in physics determining the differential calculus on spacetime.

We have considered here the bicovariant calculi on quantum groups. It is straightforward to write down the corresponding axioms for a braided group  $B$  and make a corresponding analysis. The role of the quantum double in the general theory is played by the author's double-bosonisation Hopf algebra  $B \bowtie H \ltimes B^{*\text{op}}$  in [16] (where  $H$  is the background quantum group in the braided category generated by which  $B$  lives). One should think of it as the bosonisation of the braided double  $D(B)$ , although this does not really exist by itself when the category is truly braided. Detailed diagrammatic axioms and proofs will be presented elsewhere. But the final result (as presented in Prague in June 1996) is, in suitable conventions,

**Proposition 2.8** *The possible invariant braided tangent spaces are precisely the subspaces  $L \subset \ker \epsilon \subset B^*$  which are subrepresentations under the double-bosonisation  $B \bowtie H \ltimes B^{*\text{op}}$ . Here  $B^*$  acts by the braided adjoint action,  $B$  by evaluation against projected translation and  $H$  acts as the background quantum group, cf. the canonical action in [16].*

This is the braided version of Proposition 2.2. On the other hand, when we look at the  $q$ -deformed (braided) bicovariant differential calculi on  $B = B^* = \mathbb{R}_q^n$ , we have an entirely different story when  $q \neq 1$ . In this case the double-bosonisation is the  $q$ -conformal group [17] (and the background quantum group is the dilaton-extended  $q$ -rotation group). The possible invariant braided quantum tangent spaces  $L$  are  $q$ -deformations of the projections (to vanish at zero) of irreducible subrepresentations of the conformal group acting on functions on  $\mathbb{R}^n$ . This means that  $L$  is more or less unique in this case and is (the projection of) the  $q$ -deformed set of solutions of the massless Klein-Gordon equation  $\square f = 0$ . This uniqueness gives us a clue to what will be involved in 'gluing' copies of  $\mathbb{R}_q^n$  together to define manifolds if we want to keep everything bicovariant under translation: infinite-dimensional tangent spaces defined by solutions of the massless wave equation. If we relax translational bicovariance then other  $q$ -deformations of the 1-forms on  $\mathbb{R}^n$  are also possible, including one which extends to the whole exterior algebra and which typically has the classical dimensions in each degree; see [18][1]. Some recent categorical results about exterior algebras on braided groups are in [19].

### 3 Gauge Theory

Once we have fixed choices of differential calculi on our various spaces, we can do a systematic quantum group gauge theory. This formalism has been introduced in [5], where the  $q$ -monopole over the  $q$ -deformed  $S^2$  was constructed. We work with all 'spaces' in terms of algebras. So the 'total space' of a principal bundle is an algebra  $P$ . A Hopf algebra  $A$  coacts (or  $A^*$  acts) on  $P$  and the fixed point subalgebra  $M \subseteq P$  plays the role of the 'base space' of the bundle. Among the 1-forms on  $P$  we have the horizontal 1-forms  $P\Omega^1(M)P$  and a *connection* is a choice of complement to this. One also has a theory of gauge transforms, gauge-covariant curvature,

associated vector bundles and connections on them; see [5]. The theory has been generalised to braided groups and, beyond, to systems where  $A$  is merely a coalgebra[3][2].

Here we limit ourselves to gauge theory for the case of trivial bundles of the form  $P = M \otimes A$ . In this case, all formulae look like more usual gauge theory. Thus, we consider gauge fields, curvature, gauge transformations and matter fields as

$$\alpha \in \Omega^1(M) \otimes A^*, \quad F \in \Omega^2(M) \otimes A^*, \quad \gamma \in M \otimes A^*, \quad \psi \in \Omega^n(M) \otimes V \quad (21)$$

respectively. Here  $A^*$  is assumed to act in  $V$ . At this level, we do not really need a differential calculus on  $A$  itself. We do not even need  $A$  to be a Hopf algebra. The minimum is that  $A$  should be a coalgebra, or  $A^*$  an algebra. When  $A$  is a Hopf algebra (or even a braided group) we should chose a differential calculus  $\Omega^1(A)$  and define  $\Omega^1(P) = P\Omega^1(M)P \oplus P\Omega^1(A)$ . This splitting corresponds to a canonical trivial connection on  $P = M \otimes A$ , while other connections are obtained from this by adding  $\alpha$ [5]. One can also try to restrict  $\alpha \in \Omega^1(M) \otimes L$ , although this requires some further work. None of this is needed for bare-bones gauge theory, where we work directly on the base and forget about the full geometrical structure of the principal bundle.

### 3.1 Quantum homotopy group $\pi_1(M)$

Let  $M$  and  $A^*$  be algebras. We outline the generalised gauge theory at this level, and discuss a first application. As explained, we need only  $\Omega^1, \Omega^2$  on the base algebra  $M$ . By the former, we mean a choice of first order differential calculus over  $M$  as at the start of Section 2.1. For the 2-forms we need a vector space  $\Omega^2$  such that:

1.  $\Omega^2$  is an  $M$ -bimodule.
2. There is a surjection  $\wedge : \Omega^1 \otimes_M \Omega^1 \rightarrow \Omega^2$  as  $M$ -bimodules.
3. There is a map  $d : \Omega^1 \rightarrow \Omega^2$  obeying  $d \circ d = 0$  when composed with  $d : M \rightarrow \Omega^1$ .
4. The Leibniz rule  $d(a.\omega.b) = (da) \wedge \omega.b + a.(d\omega).b - a.\omega \wedge db$  for all  $\omega \in \Omega^1$  and  $a, b \in M$ .

The first two items mean that  $\Omega^2$  is a quotient of  $\Omega^1 \otimes_M \Omega^1$ . The third and fourth conditions fully specify  $d$  once we have chosen this quotient, according to

$$d(adb) = da \wedge db, \quad d((db)a) = -db \wedge da. \quad (22)$$

In particular, we see that if (1) holds then  $\wedge$  is antisymmetric (as it is classically), but in general we cannot assume this. Also, one knows that  $\Omega^1$  can be built by quotienting a certain universal calculus on  $M$  by a bimodule. This determines likewise a natural choice of the quotient of  $\Omega^1 \otimes_M \Omega^1$  which defines  $\wedge$ . In practice, these minimal relations of  $\Omega^2$  are simply obtained

by applying  $d$  (extended by the Leibniz rule) to the relations of  $\Omega^1$ . We will use  $\Omega^2$  defined canonically from  $\Omega^1$  in this way; one could consider further quotients too.

With these basic structures fixed, we can define the essential features of gauge theory. We consider gauge fields, gauge transformations etc. as in (21) and define gauge transformations, curvature and the covariant derivative by

$$\alpha^\gamma = \gamma^{-1}\alpha\gamma + \gamma^{-1}d\gamma, \quad \psi^\gamma = \gamma^{-1}\psi, \quad F(\alpha) = d\alpha + \alpha \wedge \alpha, \quad \nabla\psi = d\psi + \alpha \wedge \psi. \quad (23)$$

Products in  $A^*$  and its action on  $V$  are understood, along with the above operations involving  $M$ . For example, the allowed gauge transformations are the invertible elements of the algebra  $M \otimes A^*$ . One can then verify the fundamental lemmas of gauge theory, namely

1.  $F(\alpha^\gamma) = \gamma^{-1}F(\alpha)\gamma$ .
2.  $\nabla^\gamma\psi^\gamma = (\nabla\psi)^\gamma$
3.  $\nabla^2\psi = F\psi$

One also has Bianchi identities, etc. at this level[5]. For example, to see exactly what is involved in the gauge covariance of  $F$ , we note first that  $d\gamma^{-1} = -\gamma^{-1}(d\gamma)\gamma^{-1}$  from the Leibniz rule. Then

$$\begin{aligned} F(\alpha^\gamma) &= d(\gamma^{-1}\alpha\gamma + \gamma^{-1}d\gamma) + (\gamma^{-1}\alpha\gamma + \gamma^{-1}d\gamma) \wedge (\gamma^{-1}\alpha\gamma + \gamma^{-1}d\gamma) \\ &= -\gamma^{-1}(d\gamma)\gamma^{-1} \wedge \alpha\gamma + \gamma^{-1}(d\alpha)\gamma - \gamma^{-1}d\gamma \wedge \gamma^{-1}d\gamma + \gamma^{-1}d^2\gamma \\ &\quad + \gamma^{-1}\alpha \wedge \alpha\gamma + \gamma^{-1}\alpha \wedge d\gamma + \gamma^{-1}d\gamma \wedge \gamma^{-1}\alpha\gamma + \gamma^{-1}d\gamma \wedge \gamma^{-1}d\gamma \end{aligned}$$

using the Leibniz rule in the generalised form for  $\Omega^2$  and the assumption that  $\wedge : \Omega^1 \otimes_M \Omega^1 \rightarrow \Omega^2$ . The 1st and 8th, 3rd and 7th, and 4th and 9th terms cancel. The 6th term vanishes by  $d^2 = 0$ . The point is that we only used the minimal structure for  $\Omega^1, \Omega^2$  as listed above. The particular conventions here are read off from [20, Fig. 3]. Interestingly, by working with  $A^{*\text{op}}$  instead of  $A^*$ , all the computations can be done diagrammatically without any transpositions or braiding, i.e. the gauge theory at this level makes sense in any monoidal category with  $\otimes$  and  $\oplus$  operations; see [20].

We are now in a position to define the ‘fundamental group of a differential algebra’  $(M, \Omega^1, \Omega^2)$ . We fix  $M \equiv (M, \Omega^1, \Omega^2)$  as the data which plays the role of a differentiable manifold in the formalism. Then for all algebras  $A^*$ , we define

$$\text{Flat}(M, A^*) = \{\alpha \in \Omega^1 \otimes A^* \mid F(\alpha) = 0\}, \quad (24)$$

the space of flat connections. The map  $A^* \rightarrow \text{Flat}(M, A^*)$  defines a covariant functor from the category of algebras to the category of sets. Functors to sets are generally representable, which

means there is essentially (up to some technical completions) an algebra  $\pi_1(M)$  such that

$$\mathrm{Hom}_{\mathrm{alg}}(\pi_1(M), A^*) \cong \mathrm{Flat}(M, A^*), \quad \forall A^* \quad (25)$$

This definition is based on the classical situation

$$\mathrm{Hom}_{\mathrm{grp}}(\pi_1(M), G) \cong \mathrm{Flat}(M, G), \quad \forall G, \quad (26)$$

where  $G$  is a Lie group,  $M$  is a manifold and we consider the holonomy on  $P = M \times G$  defined by any flat connection.

This is the basic idea, and it has many variants. It is not known at present which variant works well or how to compute  $\pi_1(M)$  in practice. As explained above, for a proper geometrical picture we need more structure on  $A^*$ . For example, a Hopf algebra structure and quantum tangent space for it. Then (25) would be defined via  $\mathrm{Hom}$  in the same category and  $\pi_1(M)$  would likewise be a Hopf algebra equipped with a choice of quantum tangent space.

While clearly sketchy, we see from these remarks that one can in principle use quantum geometry to extract genuine ‘quantum topology’ from a noncommutative algebra  $M$ . Although nontrivial computations are currently scarce, the  $q$ -monopole on  $S_q^2$  constructed in [5] likewise demonstrates nontrivial topology, although from a slightly different point of view (classically, the monopole charge is related to the fundamental group of the  $S^1$  equator of  $S^2$ ).

### 3.2 Classical Integrable Systems

In this section we specialise the above gauge theory further, to the case where  $M$  is classical. In fact, we take  $M = \mathbb{C}[\mathbf{x}]$ , the coordinate ring of  $\mathbb{R}^n$ , but we allow non-standard  $\Omega^1, \Omega^2$ . Our goal is to demonstrate the above theory. A second motivation comes from the interesting work of Muller-Hoissen[14][15], where it is already shown that even this almost classical setup can have useful applications. In effect, we combine those ideas with the more systematic gauge theory developed in [5]. For our gauge group we take the classical group with one point  $G = \{e\}$ , so  $A^* = \mathbb{C}$ . This will make the point that non-linearity emerges not from non-Abelianness of the gauge group (though one can have this too), but from the breakdown of (1).

For our first example, we take  $M = \mathbb{C}[x]$  1-dimensional and  $\Omega^1$  the 2-jet calculus defined by  $L = \{p, p^2\}$  as generated by  $c(p) = p^3/3!$  in Proposition 2.6. The relations of  $\Omega^1$  are in (17). Applying  $d$  and the Leibniz rule, we have in  $\Omega^2$ :

$$d\omega = (dx)^2, \quad \omega^2 = 0, \quad \omega \wedge dx = -dx \wedge \omega, \quad x d\omega = (d\omega)x, \quad x dx \wedge \omega = dx \wedge \omega x. \quad (27)$$

**Proposition 3.1** *Working with the 2-jet differential calculus (17),(27), the curvature of a general gauge field  $\alpha = (dx)a(x) + \omega b(x)$  is*

$$F(\alpha) = dx \wedge \omega(b' - a'' + 2a'a) - (dx)^2(a' - b - a^2).$$

The gauge transformation by  $\gamma(x)$  is

$$a \rightarrow a + \frac{\gamma'}{\gamma}, \quad b \rightarrow b - 2a\frac{\gamma'}{\gamma} + \frac{\gamma''}{\gamma} - 2\frac{(\gamma')^2}{\gamma^2},$$

under which  $F$  is invariant. This is also the gauge symmetry of the zero curvature equation  $a' = a^2 + b$ . The covariant derivative on a scalar field  $\psi$  is

$$\nabla\psi = dx(\psi' + a\psi) + \omega(\psi'' + b\psi)$$

and is covariant under  $\psi \rightarrow \frac{\psi}{\gamma}$ .

**Proof** We write out  $F = d\alpha + \alpha \wedge \alpha$  using the Leibniz rule and the relations (17),(27) to collect terms. To compute the effect of gauge transformations, we use (17) to find

$$\gamma^{-1}(dx)a\gamma = (dx)a - 2\omega a\gamma^{-1}\gamma'$$

$$\gamma^{-1}d\gamma = \gamma^{-1}(dx)\gamma' + \gamma^{-1}\omega\gamma'' = (dx)\gamma^{-1}\gamma' + \omega(\gamma^{-1}\gamma'' - 2\gamma^{-2}(\gamma')^2).$$

It is a nontrivial check to verify that  $F(\alpha^\gamma) = \gamma^{-1}F\gamma = F$  (the second equality by (27)), as it must by our general theory. The computation and covariance of  $\nabla$  is equally simple. To check  $\nabla^2\psi = F\psi$  one needs  $\nabla$  on a general 1-form field  $\sigma = (dx)s + \omega t$ , say. By similar computations to those above, this is

$$\nabla\sigma = (dx)^2(-s' + t + a^2) + dx \wedge \omega(-s'' + 2a'a - bs + at + t'). \quad (28)$$

There are Bianchi identities for  $F$  which one may check as well. All of these nontrivial facts are ensured by the gauge theory formalism in Section 3.1.  $\square$

For our second example, we take  $M = \mathbb{C}[x, y]$  and  $\Omega^1$  the 2-dimensional finite-difference calculus (20) on  $\mathbb{R}^2$  defined by  $c(p, q) = \lambda^{-2}e^{\lambda p} + \mu^{-2}e^{\mu q}$ . Applying (20) and the Leibniz rule to it, we obtain in  $\Omega^2$  the identities  $(dx)^2 = 0 = (dy)^2$  and

$$dx \wedge dy = -dy \wedge dx, \quad xdx \wedge dy = (dx \wedge dy)(x + \lambda), \quad ydx \wedge dy = (dx \wedge dy)(y + \mu) \quad (29)$$

Here,  $\Omega^2$  is 1-dimensional and has a classical form.

**Proposition 3.2** *Working with the finite-difference calculus (20),(29), the curvature of a general gauge field  $\alpha = (dx)a(x, y) + (dy)b(x, y)$  is*

$$F(\alpha) = dx \wedge dy (\partial_{p_{1,0}}b - \partial_{p_{0,1}}a + a(x, y + \mu)b - b(x + \lambda, y)a).$$

The gauge transformation by  $\gamma(x, y)$  is

$$a \rightarrow \frac{a\gamma}{\gamma(x + \lambda, y)} + \frac{1}{\lambda} \left( 1 - \frac{\gamma}{\gamma(x + \lambda, y)} \right), \quad b \rightarrow \frac{b\gamma}{\gamma(x, y + \mu)} + \frac{1}{\mu} \left( 1 - \frac{\gamma}{\gamma(x, y + \mu)} \right),$$

under which  $F \equiv (dx \wedge dy)F(x, y)$  transforms as

$$F \rightarrow F \frac{\gamma}{\gamma(x + \lambda, y + \mu)}.$$

This is also the gauge symmetry of the zero curvature equation. The covariant derivative on a scalar field  $\psi(x, y)$  is

$$\nabla\psi = dx(\partial_{p_{1,0}}\psi + a\psi) + dy(\partial_{p_{0,1}}\psi + b\psi)$$

and is covariant under  $\psi \rightarrow \frac{\psi}{\gamma}$ .

**Proof** We use the relations  $f(x, y)dx = (dx)f(x + \lambda, y)$  etc. to find the coefficient of  $dx \wedge dy$  in  $d\alpha + \alpha \wedge \alpha$ . This also gives  $\gamma^{-1}\alpha\gamma$  and  $\gamma^{-1}F\gamma$ . Finally, to compute the remaining part of the gauge transformation, we have

$$\gamma^{-1}d\gamma = \gamma^{-1}(dx)\frac{\gamma(x + \lambda, y) - \gamma(x, y)}{\lambda} + \gamma^{-1}(dy)\frac{\gamma(x, y + \mu) - \gamma(x, y)}{\mu},$$

using (20). Moving  $\gamma^{-1}$  to the right then gives the form shown. Again, it is a nontrivial check on the calculus that  $F$  is indeed covariant under this gauge transformation. The covariant derivative on a 1-form field  $\sigma = (dx)s(x, y) + (dy)t(x, y)$  is likewise computed, as

$$\nabla\sigma = dx \wedge dy (\partial_{p_{1,0}}t - \partial_{p_{0,1}}s + a(x, y + \mu)t - b(x + \lambda, y)s),$$

from which one may verify that  $\nabla^2\psi = F\psi$ . These and the other properties are all ensured by the gauge theory formalism in Section 3.1.  $\square$

Although these examples are elementary (and Proposition 3.2 is probably not new to anyone who has played with lattice gauge theory), we see that non-linear differential equations and finite-difference equations on  $\mathbb{R}^n$  can be treated as actual zero-curvature equations for a general formalism of gauge theory; just with a choice of non-standard calculi. As such, assuming trivial ‘quantum fundamental group’  $\pi_1(M)$  (as defined in Section 3.1), they are completely integrable in the sense that every solution would be of the form  $\alpha = \gamma^{-1}d\gamma$  as the gauge transform of the zero solution. It is not clear what class of integrable systems can be covered in this way, but by using the higher jet-calculi they can include arbitrary derivatives. By using a more general  $c(p)$  we have entirely new gauge theories as well. The point is that one does not need to invent and verify each of these generalised gauge theories ‘by hand’; they are instead constructed as examples of a single formalism[5][3][2].

Moreover, the formalism allows for the possibility of non-trivial gauge symmetry, which can be a group, quantum group, braided group or even a general algebra. For example, a complete formalism of lattice non-Abelian gauge theory and scalar, vector matter fields is an immediate corollary. And we are not limited to classical spaces  $M$  for the base. They can be quantum,



super or even anyonic (where the coordinate obeys  $\theta^n = 0$ ). An example of braided gauge theory on an anyonic base is worked out in detail in [2]. Finally, we are not limited to trivial bundles. Bundles can be nontrivial both in a familiar geometrical way and in a purely quantum way (even when they are geometrically trivial), the latter being controlled by a certain nonAbelian 2-cohomology; [20] is a recent review.

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## References

- [1] S. Majid. *Foundations of Quantum Group Theory*. Cambridge Univeristy Press, 1995.
- [2] S. Majid. Diagrammatics of braided group gauge theory. *Preprint* q-alg/9603018.
- [3] T. Brzezinski and S. Majid. Coalgebra gauge theory. *Preprint* q-alg/9602022.
- [4] S.L. Woronowicz. Differential calculus on compact matrix pseudogroups (quantum groups). *Commun. Math. Phys.*, 122:125–170, 1989.
- [5] T. Brzeziński and S. Majid. Quantum group gauge theory on quantum spaces. *Commun. Math. Phys.*, 157:591–638, 1993. Erratum 167:235, 1995.
- [6] K. Schmüdgen and A. Schüler. Classification of bicovariant differential calculi on quantum groups. *Commun. Math. Phys.*, 170:315–335, 1995.
- [7] S. Majid. Classification of bicovariant differential calculi. *Preprint* q-alg/9608016.
- [8] P. Hajac. Strong connections on quantum principal bundles. To appear *Commun. Math. Phys.*
- [9] B. Jurco. Differential calculus on quantized simple Lie groups. *Lett. Math. Phys.*, 22:177–186, 1991.
- [10] T. Brzeziński and S. Majid. A class of bicovariant differential calculi on Hopf algebras. *Lett. Math. Phys.*, 26:67–78, 1992.
- [11] S. Majid. Quantum and braided Lie algebras. *J. Geom. Phys.*, 13:307–356, 1994.
- [12] K. Bresser, F. Müller-Hoissen, and A. Dimakis. Non-commutative geometry of finite groups. *J. Phys. A.*, 29:2705–2735, 1996.
- [13] S. Majid. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group. *Commun. Math. Phys.*, 156:607–638, 1993.
- [14] A. Dimakis and F. Müller-Hoissen. A noncommutative differential calculus and its relation to gauge theory and gravitation. *Int. J. Mod. Phys. A (Proc. Suppl.)*, 3:374, 1992.

- [15] A. Dimakis, F. Müller-Hoissen, and T. Striker. Noncommutative differential calculus and lattice gauge theory. *J. Phys. A*, 26:1927, 1993.
- [16] S. Majid. Double bosonisation and the construction of  $U_q(g)$ , q-alg/9511001. To appear *Math. Proc. Camb. Phil. Soc.*
- [17] S. Majid. Braided geometry of the conformal algebra, q-alg/9601030. To appear *J. Math. Phys.*
- [18] S. Majid. Introduction to braided geometry and q-Minkowski space. In L. Castellani and J. Wess, editors, *Proceedings of the International School ‘Enrico Fermi’ CXXVII*, pages 267–348. IOS Press, Amsterdam, 1996.
- [19] Y. Bespalov and B. Drabant. Differential calculus in braided abelian categories. *Preprint*, preliminary.
- [20] S. Majid. Some remarks on quantum and braided group gauge theory. To appear W Pusz R. Budzynski and S. Zakrzewski, editors, *Proceedings of Quantum Groups and Quantum Spaces, 1995*. Banach Center, Warsaw.